Exercise 2.5.4

(Infinitely many solutions with the same initial condition) Show that the initial value problem $\dot{x} = x^{1/3}$, $x(0) = 0$, has an infinite number of solutions. (Hint: Construct a solution that stays at $x = 0$ until some arbitrary time t_0, after which it takes off.)

Solution

Notice that one solution to the initial value problem is $x(t) = 0$ because

$$\frac{d}{dt}(0) = (0)^{1/3} = 0$$

and $x(0) = 0$. Another one can be obtained by separating variables and integrating both sides.

$$\frac{dx}{dt} = x^{1/3}, \quad x(0) = 0$$

$$x^{-1/3}\,dx = dt$$

$$\int x^{-1/3}\,dx = \int dt$$

$$\frac{3}{2}x^{2/3} = t + C$$

Apply the initial condition $x(0) = 0$ now to determine C.

$$\frac{3}{2}(0)^{2/3} = 0 + C \quad \to \quad C = 0$$

Consequently, another solution is

$$\frac{3}{2}x^{2/3} = t$$

$$x^{2/3} = \frac{2}{3}t$$

$$x^2 = \frac{8}{27}t^3$$

$$x(t) = \pm\sqrt{\frac{8}{27}t^3}.$$

Consider a piecewise function defined by

$$x(t) = \begin{cases} 0 & \text{if } t < t_0 \\ \sqrt{\frac{8}{27}t^3} & \text{if } t > t_0 \end{cases},$$

where $t_0 > 0$ so that the initial condition is satisfied. Even though it's undefined at $t = t_0$, this function satisfies the initial value problem. Therefore, since t_0 is arbitrary, there are an infinite number of solutions.